TRINITY COLLEGE

Year 12 Mathematics Specialist Units 3, 4
Test 4 2021

Section 1 Calculator Free
Integration and Applications of Integration

STUDENT'S NAME Solutions

DATE: Tuesday 27 July **TIME:** 25 minutes **MARKS:** 25

INSTRUCTIONS:
Standard Items: Pens, pencils, drawing templates, eraser, Formula Booklet

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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1. (5 marks)

Determine the following integrals:

(a) $\int \frac{x^2-1}{x}\,dx$ [2]

$= \int x - \frac{1}{x} \, dx$

✓ algebra
✓ soln

$= \frac{x^2}{2} - \ln|x| + c$

(b) $\int \frac{\ln(x^2)}{x}\,dx$ [3]

If $y = \ln(x^2)$

$\frac{dy}{dx} = \frac{2x}{x^2}$

$= \frac{2}{x}$

$= \frac{1}{2}\int 2\,\frac{1}{x} \cdot \ln(x^2)\,dx$

$= \frac{1}{2} \cdot \frac{[\ln(x^2)]^2}{2} + c$

$= \frac{1}{4}[\ln(x^2)]^2 + c$

or

$\ln^2 x + c$

✓ factor $\frac{1}{2}$
✓ power of 2
✓ $+c$

2. (9 marks)

Determine the following integrals:

(a) $\int \frac{\sin^2\theta + \cos^2\theta}{\cos 2\theta + \sin^2\theta}\, d\theta$ [3]

$= \int \frac{1}{\cos^2\theta - \cancel{\sin^2\theta} + \cancel{\sin^2\theta}}\, d\theta$

$= \int \frac{1}{\cos^2\theta}\, d\theta$

$= \tan\theta + c$

✓ pythagorean
✓ cos 2θ
✓ soln

(b) $\int \sin^3 x\, dx$ [3]

$= \int \sin x \,.\, \sin^2 x \; dx$

$= \int \sin x \,.\, (1 - \cos^2 x)\, dx$

$= \int \sin x - \sin x [\cos x]^2 \; dx$

$= -\cos x + \frac{[\cos x]^3}{3} + c$

✓ pythagoras
✓ expand
✓ soln

(c) $\int \frac{x^2}{x-1} dx$ [3]

$$\begin{array}{r|l} & x + 1 \\ x-1 & x^2 + 0x + 0 \\ & x^2 - x \\ & x + 0 \\ & x - 1 \\ & 1 \end{array}$$

$= \int x + 1 + \frac{1}{x-1} \; dx$

$= \frac{x^2}{2} + x + \ln|x-1| + c$

✓ division
✓ $\int x+1$
✓ $\int \frac{1}{x-1}$

3. (5 marks)

(a) Express $\dfrac{x+7}{(x+1)(x-2)}$ in the form $\dfrac{a}{x+1}+\dfrac{b}{x-2}$. [2]

$\Rightarrow \quad x+7 = (x-2)a + (x+1)b$

If $x=2 \Rightarrow 9 = 3b$, $\therefore b = 3$ ✓ a

If $x=-1 \Rightarrow 6 = -3a$, $\therefore a = -2$ ✓ b

So $\dfrac{x+7}{(x+1)(x-2)} = \dfrac{-2}{x+1} + \dfrac{3}{x-2}$

(b) Hence, determine $\int \dfrac{x+7}{(x+1)(x-2)}dx$ [3]

$= \int \dfrac{-2}{x+1} + \dfrac{3}{x-2} \, dx$ ✓ partial

$= -2\ln|x+1| + 3\ln|x-2| + c$ ✓ $-2\ln|x+1|$

✓ $3\ln|x-2|$

4. (6 marks)

Evaluate exactly: $\int_0^{\sqrt{2}} \sqrt{1-\frac{x^2}{4}}\,dx$ using the substitution $x = 2\sin\theta$

$$= \int_0^{\pi/4} \sqrt{1 - \frac{4\sin^2\theta}{4}} \,.\, 2\cos\theta \, d\theta$$

$$= \int_0^{\pi/4} \cos\theta \,.\, 2\cos\theta \, d\theta$$

$$= \int_0^{\pi/4} 2\cos^2\theta \, d\theta$$

$$= \int_0^{\pi/4} \cos 2\theta + 1 \, d\theta$$

$$= \left[\frac{1}{2}\sin 2\theta + \theta \right]_0^{\pi/4}$$

$$= \left(\frac{1}{2}\sin\frac{\pi}{2} + \frac{\pi}{4} \right) - \left(-\frac{1}{2}\sin 0 + 0 \right)$$

$$= \frac{1}{2} + \frac{\pi}{4}$$

$x = 2\sin\theta$

$dx = 2\cos\theta \, d\theta$

Boundaries

$\sqrt{2} = 2\sin\theta$

$\Rightarrow \theta = \pi/4$

$0 = 2\sin\theta$

$\Rightarrow \theta = 0$

✓ $dx/d\theta$

✓ boundaries

✓ simplification of √

✓ double angle

✓ integral

✓ soln

TRINITY COLLEGE

Year 12 Mathematics Specialist Units 3, 4
Test 4 2021

Section 2 Calculator Assumed

Integration and Applications of Integration

STUDENT'S NAME ______________________________

DATE: Tuesday 27 July **TIME:** 25 minutes **MARKS:** 25

INSTRUCTIONS:

Standard Items: Pens, pencils, drawing templates, eraser

Special Items: Three calculators, notes on one side of a single A4 page (these notes to be handed in with this assessment)

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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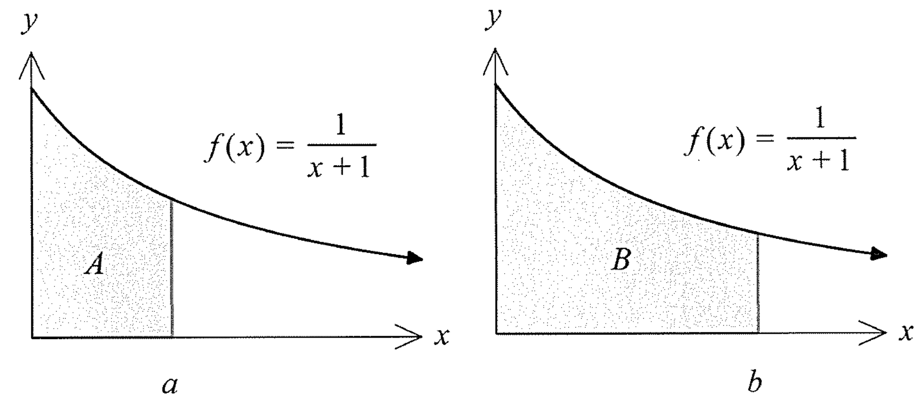
5. (4 marks)

The area labelled B is three times the area labelled A .



Express b in terms of a.

So $3A = B$ ✓ eqn

$\Rightarrow 3\int_0^a \frac{1}{x+1}\,dx = \int_0^b \frac{1}{x+1}\,dx$ ✓ integral

$\Rightarrow 3[\ln|x+1|]_0^a = [\ln|x+1|]_0^b$ ✓ eqn with a + b

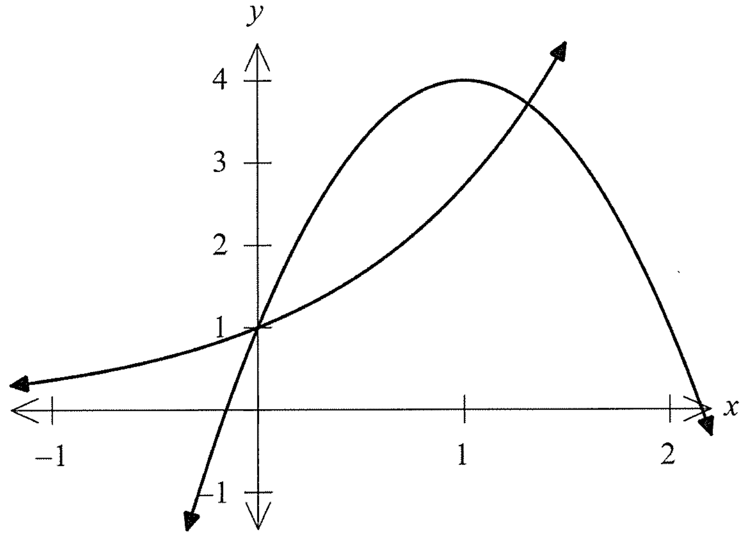
$\Rightarrow \ln(a+1)^3 = \ln(b+1)$ ✓ eqn b =

$\Rightarrow (a+1)^3 = b+1$

So $b = (a+1)^3 - 1$

6. (8 marks)

Consider the two functions $f(x) = e^x$ and $g(x) = -3x^2 + 6x + 1$.



(a) (i) Write an integral expression for the approximate enclosed area between the curves. [2]

Point of intersection $(0, 1)$ and $(1.31, 3.71)$

$$\text{Area} = \int_0^{1.31} -3x^2 + 6x + 1 - e^x \, dx$$

✓ $x = 1.31$
✓ area integral

(ii) Calculate the approximate enclosed area. [2]

Area = 1.50 units^2

✓ value
✓ units^2

(b) (i) Write down an integral expression for volume formed when the enclosed region is rotated about the x-axis. [2]

$$\text{Vol} = \pi \int_0^{1.31} (-3x^2 + 6x + 1)^2 - (e^x)^2 \, dx$$

✓ π
✓ eqn

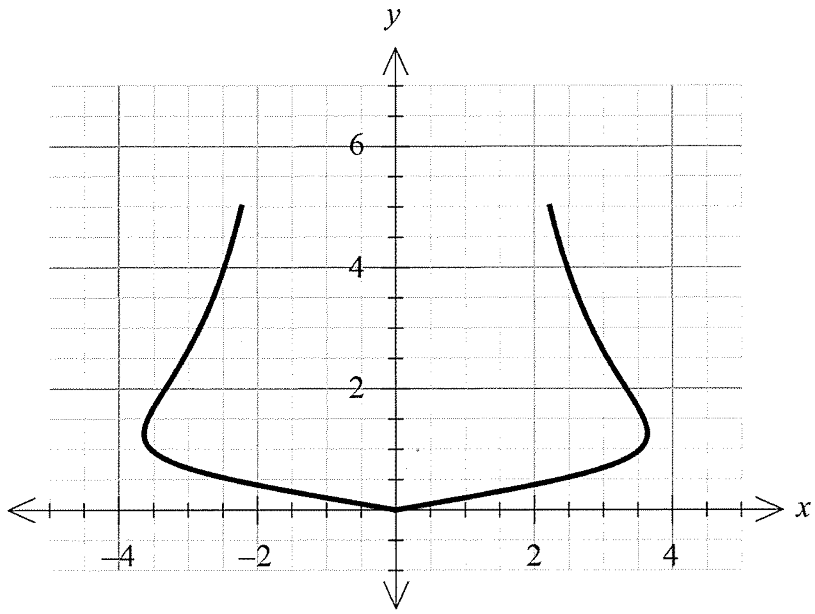
(ii) Calculate the volume formed when the enclosed region is rotated about the x-axis. [2]

Vol = 25.63 units^3

✓ value
✓ units^3

7. (5 marks)

The top part of a wine glass is modelled by rotating the graph of $x^2 = y^2(25 - x^2 y)$ from $y = 0$ to $y = 5$ about the y axis as shown below. Dimensions are measured in centimetres.



Calculate, correct to the nearest 0.01 cm, the depth of wine in the glass if it is to contain 75% of its maximum volume.

$$Vol_{max} = \pi \int_0^5 [f(y)]^2 \, dy$$

$$= \pi \int_0^5 \frac{25y^2}{1+y^3} \, dy$$

$$= 126.61 \text{ cm}^3$$

So, 75% of this volume

$$\Rightarrow 0.75 \times 126.61 = \pi \int_0^h \frac{25y^2}{1+y^3} \, dy$$

Solving $h = 3.32$ cm

Now

$$x^2 = y^2(25 - x^2 y)$$

$$x^2 = 25y^2 - x^2 y^3$$

$$x^2(1+y^3) = 25y^2$$

$$x^2 = \frac{25y^2}{1+y^3}$$

✓ eqn x^2

✓ vol eqn

✓ 126.61

✓ 75% =

✓ h =

8. (8 marks)

The table below gives the value of a function obtained from an experiment.

x	0	1	2	3	4	5	6
$f(x)$	9.3	9.0	8.3	6.5	2.3	-7.6	-10.5

Two different methods are used to approximate $\int_0^6 f(x)\,dx$.

(a) Method 1: Using three equal subintervals, estimate $\int_0^6 f(x)\,dx$ by using trapeziums. [4]

$w = 2$

$T_1 = \frac{1}{2}(9.3 + 8.3) \times 2 = 17.6$ ✓

$T_2 = \frac{1}{2}(8.3 + 2.3) \times 2 = 10.6$ ✓

$T_3 = \frac{1}{2}(2.3 + -10.5) \times 2 = -8.2$ ✓

$\therefore$ Approx is $17.6 + 10.6 - 8.2 = 20$ ✓

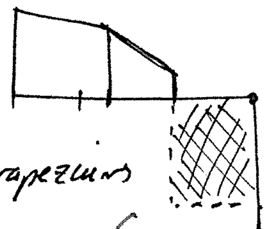
(b) Method 2: The function $g(x) = 0.14x^4 - 1.57x^3 + 4.63x^2 - 4.34x + 9.48$ is used to estimate $f(x)$

x	0	1	2	3	4	5	6
$f(x)$	9.3	9.0	8.3	6.5	2.3	-7.6	-10.5
$g(x)$	9.48	8.34	9	7.08	1.56	-5.22	-7.56

Calculate $\int_0^6 g(x)\,dx$ $= 21.168$ ✓ [1]

(c) For this question, explain the limitations of each method and comment on which estimate is more accurate. [3]

Method 1:



limitation
– only 3 trapeziums ✓

If we use trapeziums of width 1 unit

Area $= \frac{1}{2}(f(0) + 2f(1) + \ldots + f(6)) \times 1$

$= 17.75$

Method 2:
limitation
– fn has errors for each value ✓

$\therefore$ Method 1 is more accurate. ✓